On the unification of classical and novel integrable surfaces: II. Difference geometry

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Abstract

A novel class of discrete integrable surfaces is recorded. This class of discrete O surfaces is shown to include discrete analogues of classical surfaces such as isothermic, 'linear' Weingarten, Guichard and Petot surfaces. Moreover, natural discrete analogues of the Gaußian and mean curvatures for surfaces parametrized in terms of curvature coordinates are used to define surfaces of constant discrete Gaußian and mean curvatures and discrete minimal surfaces. Remarkably, these turn out to be proto-typical examples of discrete O surfaces. It is demonstrated that the construction of a Bäcklund transformation for discrete O surfaces leads in a natural manner to an associated parameter-dependent linear representation. Canonical discretizations of the classical pseudosphere and breather pseudospherical surfaces are generated. Connections with pioneering work by Bobenko and Pinkall are established.

1 Introduction

In Schief & Konopelchenko (2000), a novel class of integrable surfaces has been introduced and shown to include as canonical members a variety of classical surfaces such as isothermic, constant mean curvature, minimal, 'linear' Weingarten, Guichard and Petot surfaces and surfaces of constant Gaußian curvature. The definition of this class of O surfaces is entirely based on the classical notions of conjugate and curvature coordinates. Both coordinate systems possess natural difference-geometric counterparts which have been widely used in the construction of integrable discrete geometries (Bobenko & Seiler 1999). It turns out that these may indeed be used to define discrete O surfaces. In particular, difference-geometric analogues of the above-mentioned classical surfaces are readily constructed. Thus, the formalism developed in this paper may be regarded as a first step towards a unified description of integrability-preserving discretizations of differential geometries.

Here, we consider two-dimensional lattices (discrete surfaces) in a Euclidean space $\mathbb{R}^3$ which consist of planar quadrilaterals. These conjugate lattices may be mapped to parallel conjugate lattices by means of the discrete Combescure transformation (Konopelchenko & Schief 1998). Moreover, if a conjugate lattice is discrete orthogonal, that is the quadrilaterals are inscribed in circles (Bobenko

& Seiler 1999), then the lattice gives rise to a one-parameter family of parallel lattices on the unit sphere (Konopelchenko & Schief 1998). As an important preliminary, we exploit the existence of these 'spherical representations' to define in a geometric and algebraic manner discrete Gaußian and mean curvatures for curvature lattices. Sets of n parallel conjugate lattices $\mathbb{R}^3$ are then canonically associated with three parallel conjugate lattices in a dual (pseudo-)Euclidean space $\mathbb{R}^n$. If the dual lattices are also discrete orthogonal then the surfaces in $\mathbb{R}^3$ and $\mathbb{R}^n$ are termed (dual) discrete O surfaces.

On appropriate specification of the dimension and the metric of the dual space, important examples of discrete O surfaces may be isolated. Thus, discrete isothermic surfaces are obtained which, remarkably, turn out to be precisely those proposed by Bobenko & Pinkall (1996). Moreover, we show that curvature lattices of constant discrete mean curvature likewise constitute discrete O surfaces. In fact, discrete constant mean curvature and mininal surfaces turn out to be particular discrete isothermic surfaces. It is also demonstrated that any discrete constant mean curvature surface may be associated with a second parallel discrete constant mean curvature surface and a parallel surface of constant postive discrete Gaussian curvature, the latter being another O surface. This result may be interpreted as the analogue of a classical theorem due to Bonnet and is in agreement with that set down in Bobenko & Pinkall (1999).

Discretizations of Guichard, 'linear' Weingarten and Petot surfaces are also recorded. The discrete Petot surfaces are shown to coincide with those defined and studied in Schief (1997). A Bäcklund transformation for discrete O surfaces is obtained by constraining the discrete analogue of the classical Fundamental transformation (Konopelchenko & Schief 1998) in such a way that the discrete orthogonality conditions are preserved. As a by-product, a matrix Lax pair for discrete O surfaces is derived. As an application of the Bäcklund transformation for discrete O surfaces, discretizations of the classical pseudosphere and breather pseudospherical surfaces are generated.

2 Conjugate lattices and the discrete Combescure transformation

In the following, we are concerned with the geometry of two-dimensional lattices in a three-dimensional Euclidean space, that is maps

$$\vec{R} : \mathbb{Z}^2 \to \mathbb{R}^3, \quad (n_1, n_2) \mapsto \vec{r}(n_1, n_2). \tag{1}$$

These may be regarded as difference-geometric analogues or *discretizations* of surfaces in $\mathbb{R}^3$ and are therefore referred to as *discrete surfaces* (Bobenko & Seiler 1999). If the position vector $\vec{R}$ to a discrete surface $\Sigma \subset \mathbb{R}^3$ obeys a 'hyperbolic' linear difference equation of the form

$$\vec{R}_{(12)} - \vec{R} = a(\vec{R}_{(1)} - \vec{R}) + b(\vec{R}_{(2)} - \vec{R}), \tag{2}$$

where the notation

$$\begin{aligned} \vec{R} &= \vec{R}(n_1, n_2), & \vec{R}_{(12)} &= \vec{R}(n_1+1, n_2+1) \\ \vec{R}_{(1)} &= \vec{R}(n_1+1, n_2), & \vec{R}_{(2)} &= \vec{R}(n_1, n_2+1) \end{aligned} \tag{3}$$

has been adopted, then the lattice is termed *conjugate* (Bobenko & Seiler 1999). In geometric terms, this algebraic condition is equivalent to the requirement that the quadrilaterals $<\vec{R}, \vec{R}_{(1)}, \vec{R}_{(2)}, \vec{R}_{(12)}>$ of the lattice Σ be planar. In this case, it is natural to introduce the decomposition

$$\vec{R}_{(1)} - \vec{R} = \vec{X}H, \quad \vec{R}_{(2)} - \vec{R} = \vec{Y}K. \tag{4}$$

Since the 'tangent vectors' $\vec{X}$ and $\vec{Y}$ are only defined up to their moduli, one is at liberty to choose a convenient 'gauge'. Indeed, we here repair to the gauge employed in Konopelchenko & Schief (1998). Thus, the compatibility condition for the relations (4) is readily shown to lead to the linear systems

$$\begin{aligned} \vec{X}_{(2)} &= \frac{\vec{X} + q\vec{Y}}{\Gamma}, & H_{(2)} &= \frac{H + pK}{\Gamma} \\ \vec{Y}_{(1)} &= \frac{\vec{Y} + p\vec{X}}{\Gamma}, & K_{(1)} &= \frac{K + qH}{\Gamma}, \end{aligned} \tag{5}$$

where Γ is defined by

$$\Gamma^2 = 1 - pq \tag{6}$$

and the functions p and q are related to a and b by

$$a = \frac{H_{(2)}}{\Gamma H}, \quad b = \frac{K_{(1)}}{\Gamma K}. \tag{7}$$

The system $(5)_{2,4}$ may be regarded as *adjoint* to the linear system $(5)_{1,3}$.

Conversely, if $\{\vec{X}, \vec{Y}, H, K\}$ constitutes a solution of the linear systems (5) for some functions p and q then the relations (4) are compatible and $\vec{R}$ may be interpreted as the position vector of a conjugate lattice $\Sigma \subset \mathbb{R}^3$. A second solution $\{H_*, K_*\}$ of the adjoint system $(5)_{2,4}$ gives rise to a second conjugate lattice Σ_*, the position vector of which is defined by

$$\vec{R}_{*(1)} - \vec{R}_* = \vec{X}H_*, \quad \vec{R}_{*(2)} - \vec{R}_* = \vec{Y}K_*. \tag{8}$$

Accordingly, corresponding edges of the quadrilaterals $<\vec{R}, \vec{R}_{(1)}, \vec{R}_{(2)}, \vec{R}_{(12)}>$ and $<\vec{R}_*, \vec{R}_{*(1)}, \vec{R}_{*(2)}, \vec{R}_{*(12)}>$ are parallel and hence the quadrilaterals themselves are parallel. The lattice Σ_* is termed a *discrete Combescure transform* (Konopelchenko & Schief 1998) of the lattice Σ. Thus, the discrete Combescure transformation maps conjugate lattices to *parallel* conjugate lattices.

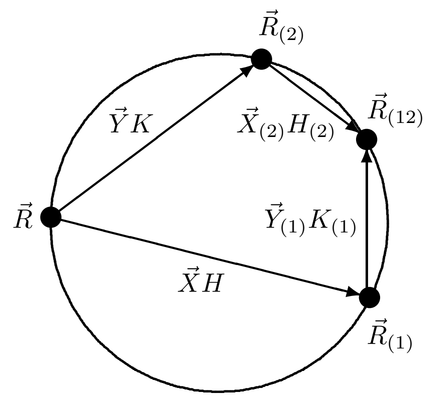

Figure 1: An embedded quadrilateral of a curvature lattice

3 Curvature lattices and discrete Gaußian and mean curvatures

Conjugate lattices are said to be *curvature lattices* if they are, in addition, *discrete orthogonal*[1] (Bobenko & Seiler 1999), that is if the quadrilaterals are inscribed in circles. Analytically, curvature lattices are characterized by the following three equivalent conditions:

$$\begin{aligned} &\text{(i)} && \vec{X}_{(2)}\cdot\vec{Y}+\vec{Y}_{(1)}\cdot\vec{X}=0\\ &\text{(ii)} && 2\vec{X}\cdot\vec{Y}+p\vec{X}^2+q\vec{Y}^2=0\\ &\text{(iii)} && \vec{X}_{(2)}^2=\vec{X}^2,\quad \vec{Y}_{(1)}^2=\vec{Y}^2. \end{aligned} \tag{9}$$

In fact, since the tangent vectors $\vec{X}$ and $\vec{Y}$ of conjugate lattices are oriented in such a way that

$$\vec{X}_{(2)}\times\vec{Y}+\vec{Y}_{(1)}\times\vec{X}=0, \tag{10}$$

combination of the conditions (i) and (iii) shows that opposite angles in any quadrilateral are either equal or add up to π according to whether the edges intersect or not. This proves that the quadrilaterals are inscribed in circles. An ‘embedded’ quadrilateral is displayed in Figure 1. It is also noted that the condition (iii) implies that we may assume without loss of generality that $\vec{X}$ and $\vec{Y}$ constitute unit vectors. Indeed, it is readily verified that appropriate scaling of $\vec{X},\vec{Y},H,K$ and p,q leads to

$$\vec{X}^2=1,\quad \vec{Y}^2=1. \tag{11}$$

This normalization will be adopted throughout the paper.

It is evident that the Combescure transformation maps within the class of curvature lattices since the discrete orthogonality conditions (9) do not involve

[1]It is emphasized that discrete orthogonality is only defined in conjunction with conjugacy.

the solutions $\{H, K\}$ of the adjoint system $(5)_{2,4}$. This implies that any curvature lattice may be mapped via a Combescure transformation onto the unit sphere $\mathbb{S}^2$. In fact, as pointed out in Konopelchenko & Schief (1998), there exists a one-parameter family of curvature lattices $\Sigma_\circ$ which are parallel to any given curvature lattice. These are constructed by choosing an arbitrary point $\vec{R}_\circ(0,0) \in \mathbb{S}^2$ corresponding to the vertex $\vec{R}(0,0)$ of the curvature lattice Σ and successively drawing lines parallel to the edges of Σ, thereby identifying the points of intersection with the unit sphere as the vertices of the parallel lattice $\Sigma_\circ$. It is natural to refer to this family of parallel curvature lattices as *spherical representations* of the curvature lattice Σ. Each spherical representation $\Sigma_\circ$ corresponds to a particular solution $\{H_\circ, K_\circ\}$ of the adjoint system

$$H_{\circ(2)} = \frac{H_\circ + pK_\circ}{\Gamma}, \quad K_{\circ(1)} = \frac{K_\circ + qH_\circ}{\Gamma}. \tag{12}$$

We are now in a position to define discrete Gaußian and mean curvatures for conjugate lattices. Thus, let $<\vec{R}, \vec{R}_{(1)}, \vec{R}_{(2)}, \vec{R}_{(12)}>$ be an embedded quadrilateral of a curvature lattice Σ. The area A_u of the 'upper' triangle $<\vec{R}, \vec{R}_{(2)}, \vec{R}_{(12)}|$ is then readily shown to be

$$A_u = |H_{(2)}K|\Xi, \quad \Xi = \frac{\sqrt{1 - (\vec{X}\cdot\vec{Y})^2}}{2\Gamma} \tag{13}$$

while the area A_l of the 'lower' triangle $<\vec{R}, \vec{R}_{(1)}, \vec{R}_{(12)}|$ reads

$$A_l = |K_{(1)}H|\Xi. \tag{14}$$

The total area of the quadrilateral is given by

$$A = |H_{(2)}K + K_{(1)}H|\Xi. \tag{15}$$

In a similar manner, one obtains for the total area of the parallel quadrilateral $<\vec{R}_\circ, \vec{R}_{\circ(1)}, \vec{R}_{\circ(2)}, \vec{R}_{\circ(12)}>$ of a spherical representation $\Sigma_\circ$ the expression

$$A_\circ = |H_{\circ(2)}K_\circ + K_{\circ(1)}H_\circ|\Xi. \tag{16}$$

In analogy to the differential-geometric case (Eisenhart 1960) in which the Gaußian curvature of a surface Σ is defined as the inverse ratio of the areas of an infinitesimal surface element and its spherical representation, that is

$$\mathsf{K} = \frac{(\vec{N}_x \times \vec{N}_y)\cdot\vec{N}}{(\vec{R}_x \times \vec{R}_y)\cdot\vec{N}} = \pm\frac{|\vec{N}_x \times \vec{N}_y|}{|\vec{R}_x \times \vec{R}_y|}, \tag{17}$$

where x, y and $\vec{N}$ denote curvature coordinates and the unit normal to Σ respectively, we now propose the following:[2]

[2]The sign in (18) is chosen in such a way that the classical expression for the Gaußian curvature is obtained in the natural continuum limit.

Definition 1 (Discrete Gaußian curvature) *The* discrete Gaußian curvature *of a curvature lattice* Σ *with respect to an associated spherical representation* $\Sigma_\circ$ *is defined by*

$$\mathsf{K} = \frac{H_{\circ(2)}K_\circ + K_{\circ(1)}H_\circ}{H_{(2)}K + K_{(1)}H}. \tag{18}$$

It is noted that, by construction, the above definition of the discrete Gaußian curvature is 'geometric' in the sense that it is invariant under a re-labelling of the curvature lattice. Indeed, it is readily verified that K may also be brought into the form

$$\mathsf{K} = \frac{H_{\circ(2)}K_{\circ(1)} + H_\circ K_\circ}{H_{(2)}K_{(1)} + HK}. \tag{19}$$

The mean curvature of a surface Σ parametrized in terms of curvature coordinates is given by

$$\mathsf{M} = -\frac{(\vec{N}_x \times \vec{R}_y)\cdot\vec{N} + (\vec{R}_x \times \vec{N}_y)\cdot\vec{N}}{(\vec{R}_x \times \vec{R}_y)\cdot\vec{N}} = -\frac{\pm|\vec{N}_x \times \vec{R}_y| \pm |\vec{R}_x \times \vec{N}_y|}{|\vec{R}_x \times \vec{R}_y|}, \tag{20}$$

where the signs in the second expression for M have to be chosen appropriately. Since the expression $|\vec{N}_x \times \vec{R}_y|/2$ is nothing but the area of the triangle spanned by the vectors $\vec{N}_x$ and $\vec{R}_y$, its canonical difference-geometric counterpart could be any of the areas

$$|H_{\circ(2)}K|\Xi, \quad |K_{(1)}H_\circ|\Xi, \quad |H_{\circ(2)}K_{(1)}|\Xi, \quad |H_\circ K|\Xi. \tag{21}$$

Analogously, the area $|\vec{R}_x \times \vec{N}_y|/2$ of the triangle spannned by the vectors $\vec{R}_x$ and $\vec{N}_y$ gives rise to the discretizations

$$|H_{(2)}K_\circ|\Xi, \quad |K_{\circ(1)}H|\Xi, \quad |H_{(2)}K_{\circ(1)}|\Xi, \quad |HK_\circ|\Xi. \tag{22}$$

However, any potential candidate for the discrete mean curvature expressed in terms of the above areas must prove 'geometric'. It turns out there indeed exists a 'symmetric' expression which satisfies this requirement. Thus, the following definition is invariant under a re-labelling of the curvature lattice:

Definition 2 (Discrete mean curvature) *The* discrete mean curvature *of a curvature lattice* Σ *with respect to an associated spherical representation* $\Sigma_\circ$ *is defined by*

$$\mathsf{M} = -\frac{H_{\circ(2)}K + K_{(1)}H_\circ + H_{(2)}K_\circ + K_{\circ(1)}H}{H_{(2)}K + K_{(1)}H}. \tag{23}$$

We conclude this section with a natural definition of discrete analogues of classical surfaces of constant Gaußian and mean curvature and minimal surfaces:

Definition 3 (Surfaces of constant discrete Gaußian and mean curvature and discrete minimal surfaces) *A curvature lattice* Σ *constitutes* (i)

a surface of constant discrete Gaußian curvature, (ii) *a* surface of constant discrete mean curvature, (iii) *a* discrete minimal surface *if there exists an associated spherical representation* $\Sigma_\circ$ *such that*

$$\text{(i)}\quad \mathsf{K} = \text{const}, \quad \text{(ii)}\quad \mathsf{M} = \text{const}, \quad \text{(iii)}\quad \mathsf{M} = 0 \tag{24}$$

respectively.

It is remarkable that these discrete surfaces prove integrable since they constitute canonical members of the class of discrete O surfaces to be introduced in the sequel. In fact, in the case of surfaces of constant positive discrete Gaußian and constant discrete mean curvature and discrete minimal surfaces, they coincide with those proposed by Bobenko & Pinkall (1996, 1999).

4 Parallel conjugate lattices and their duals

As in the differential-geometric case (Schief & Konopelchenko 2000), we now investigate the geometric and algebraic properties of sets $\{\Sigma_1, \ldots, \Sigma_n\}$ of discrete surfaces which are related by discrete Combescure transformations. To this end, we consider the linear systems

$$\begin{aligned} \vec{X}_{(2)} &= \frac{\vec{X} + q\vec{Y}}{\Gamma}, \quad \underline{H}_{(2)} = \frac{\underline{H} + p\underline{K}}{\Gamma} \\ \vec{Y}_{(1)} &= \frac{\vec{Y} + p\vec{X}}{\Gamma}, \quad \underline{K}_{(1)} = \frac{\underline{K} + q\underline{H}}{\Gamma}, \end{aligned} \tag{25}$$

where $\vec{X}, \vec{Y} \in \mathbb{R}^3$ and $\underline{H}, \underline{K} \in \mathbb{R}^n$ are interpreted as column and row vectors respectively, and define a matrix $\underline{\vec{R}} \in \mathbb{R}^{3,n}$ via the compatible equations

$$\underline{\vec{R}}_{(1)} - \underline{\vec{R}} = \vec{X}\underline{H}, \quad \underline{\vec{R}}_{(2)} - \underline{\vec{R}} = \vec{Y}\underline{K}. \tag{26}$$

Here, the function Γ is defined, as usual, by $\Gamma^2 = 1 - pq$. Thus, the geometric interpretation given below is immediate:

The vectors

$$\vec{R}_\kappa \in \mathbb{R}^3, \quad \kappa = 1, \ldots, n$$

parametrize parallel conjugate lattices $\Sigma_\kappa \subset \mathbb{R}^3$ *with tangent vectors* $\vec{X}$ *and* $\vec{Y}$.

However, since there exists complete symmetry between $\{\vec{X}, \vec{Y}\}$ and $\{\underline{H}, \underline{K}\}$ and the definition of conjugate lattices is in fact independent of the dimension of the ambient space, the following point of view is also valid:

The vectors

$$\underline{R}^k \in \mathbb{R}^n, \quad k = 1, 2, 3$$

parametrize parallel conjugate lattices $\Sigma^k \subset \mathbb{R}^n$ *with tangent vectors* $\underline{H}$ *and* $\underline{K}$.

We refer to the discrete surfaces Σ^k as *dual* to the discrete surfaces Σ_κ. The concept of dual conjugate lattices has been exploited in the context of integrable

difference geometries by several authors (Konopelchenko & Schief 1998; Doliwa & Santini 1999). As mentioned earlier, we here regard the ambient space $\mathbb{R}^3$ as a Euclidean space even though the generalization to pseudo-Euclidean spaces $\mathbb{R}^3$ and their higher-dimensional analogues is straightforward. By contrast, it turns out pivotal to deal with pseudo-Euclidean dual spaces $\mathbb{R}^n$. Thus, we endow $\mathbb{R}^n$ with the inner product

$$\underline{H} \cdot \underline{K} = \underline{H}\,\underline{K}^{\intercal} = \sum_{\kappa,\mu=1}^{n} H_\kappa S^{\kappa\mu} K_\mu, \tag{27}$$

where $S = (S^{\kappa\mu})$ is a constant symmetric matrix.

5 A novel class of discrete integrable surfaces

5.1 The geometry of discrete O surfaces

Since the discrete Combescure transformation maps within the class of curvature lattices, it is natural to focus on parallel conjugate lattices Σ^k which are dual to a set of parallel curvature lattices Σ_κ. In the generic case, these are not necessarily discrete orthogonal. In fact, the discrete orthogonality condition

$$\underline{H}_{(2)} \cdot \underline{K} + \underline{K}_{(1)} \cdot \underline{H} = 0 \tag{28}$$

or, equivalently,

$$\underline{H}_{(2)}^2 = \underline{H}^2, \quad \underline{K}_{(1)}^2 = \underline{K}^2 \tag{29}$$

imposes severe constraints on the discrete surfaces Σ_κ. Thus, imposition of the discrete orthogonality condition on the dual lattices leads to a natural discretization of the integrable class of *O surfaces* recorded in Schief & Konopelchenko (2000).

Definition 4 (Discrete O surfaces) *Parallel conjugate lattices* $\Sigma_\kappa \subset \mathbb{R}^3$ *and their duals* $\Sigma^k \subset \mathbb{R}^n$ *are termed* (dual) discrete O surfaces *if both the lattices* Σ_κ *and* Σ^k *are discrete orthogonal.*

Before we establish the integrability of discrete O surfaces by deriving a parameter-dependent linear representation and an associated Bäcklund transformation, we demonstrate below how discrete versions of classical surfaces such as isothermic, constant mean curvature, minimal, 'linear' Weingarten, Guichard and Petot surfaces and surfaces of constant Gaußian curvature may be retrieved as canonical examples of discrete O surfaces. The significance of the discrete Gaußian and mean curvatures defined in §3 is thereby brought to light.

5.2 Examples

5.2.1 Surfaces of constant discrete Gaußian curvature

As in Schief & Konopelchenko (2000), we begin with the simplest choice

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{30}$$

corresponding to a two-dimensional Euclidean dual space $\mathbb{R}^2$. In this case, the discrete orthogonality condition (28) takes the form

$$H_{1(2)}K_1 + K_{1(1)}H_1 + H_{2(2)}K_2 + K_{2(1)}H_2 = 0. \tag{31}$$

By virtue of (18), this is equivalent to the requirement that the discrete Gaußian curvatures of Σ_1 and Σ_2 with respect to one and hence any spherical representation $\Sigma_\circ$ be related by

$$\mathsf{K}_1 = -\mathsf{K}_2. \tag{32}$$

Alternatively, if we consider a pseudo-Euclidean dual space $\mathbb{R}^2$ with

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{33}$$

then the discrete orthogonality condition reads

$$H_{1(2)}K_1 + K_{1(1)}H_1 = H_{2(2)}K_2 + K_{2(1)}H_2 \tag{34}$$

so that

$$\mathsf{K}_1 = \mathsf{K}_2. \tag{35}$$

We therefore conclude that a pair of parallel curvature lattices constitute discrete O surfaces if the discrete Gaußian curvatures of corresponding parallel quadrilaterals are of the same magnitude. In particular, if we confine the lattice Σ_2 to the sphere with $\mathsf{K}_2 = 1$ then the discrete surface Σ_1 is of constant discrete Gaußian curvature with respect to the spherical representation $\Sigma_\circ = \Sigma_2$. Since in the differential-geometric setting (Schief & Konopelchenko 2000) the above analysis has been shown to lead to classical (pseudo)spherical surfaces, it has been established that, remarkably, the natural discrete analogues of surfaces of constant Gaußian curvature coincide with the surfaces of discrete constant Gaußian curvature defined in §3.

5.2.2 Discrete isothermic and minimal surfaces

The choice

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{36}$$

leads to the discrete orthogonality condition

$$H_{1(2)}K_2 + K_{1(1)}H_2 + H_{2(2)}K_1 + K_{2(1)}H_1 = 0 \tag{37}$$

and the alternative characterization (29) yields

$$H_1 H_2 = \alpha(n_1), \quad K_1 K_2 = -\beta(n_2). \tag{38}$$

The latter may be used to eliminate the quantities H_2 and K_2 in (37) to obtain

$$(\beta H_{1(2)} H_1 - \alpha K_{1(1)} K_1)(K_{1(1)} H_{1(2)} + H_1 K_1) = 0. \tag{39}$$

If we now assume that the quadrilaterals are embedded and non-degenerate then the second factor in (39) is non-vanishing and whence

$$\frac{H_{1(2)} H_1}{K_{1(1)} K_1} = \frac{\alpha}{\beta} = \frac{H_{2(2)} H_2}{K_{2(1)} K_2} \tag{40}$$

with $\alpha\beta > 0$. In terms of the edges of the quadrilaterals, these relations translate into

$$\frac{|\vec{R}_{1(12)} - \vec{R}_{1(2)}||\vec{R}_{1(1)} - \vec{R}_1|}{|\vec{R}_{1(12)} - \vec{R}_{1(1)}||\vec{R}_{1(2)} - \vec{R}_1|} = \frac{\alpha}{\beta} = \frac{|\vec{R}_{2(12)} - \vec{R}_{2(2)}||\vec{R}_{2(1)} - \vec{R}_2|}{|\vec{R}_{2(12)} - \vec{R}_{2(1)}||\vec{R}_{2(2)} - \vec{R}_2|}. \tag{41}$$

Curvature lattices with *cross-ratios* of the form (41) have been termed *discrete isothermic* by Bobenko & Pinkall (1996). Thus, the discrete surfaces Σ_1 and Σ_2 constitute *discrete isothermic surfaces* which are related by the *discrete Christoffel transformation* (Bobenko & Pinkall 1996)

$$\vec{R}_{2(1)} - \vec{R}_2 = \alpha \frac{\vec{R}_{1(1)} - \vec{R}_1}{(\vec{R}_{1(1)} - \vec{R}_1)^2}, \quad \vec{R}_{2(2)} - \vec{R}_2 = -\beta \frac{\vec{R}_{1(2)} - \vec{R}_1}{(\vec{R}_{1(2)} - \vec{R}_1)^2}. \tag{42}$$

If we identify Σ_2 with a spherical representation of Σ_1 then $H_2 = H_\circ$ and $K_2 = K_\circ$ which, in turn, implies that the discrete surface Σ_1 is discrete minimal since

$$\mathsf{M}_1 = -\frac{H_{2(2)} K_1 + K_{1(1)} H_2 + H_{1(2)} K_2 + K_{2(1)} H_1}{H_{1(2)} K_1 + K_{1(1)} H_1} = 0 \tag{43}$$

by virtue of (37). Thus, the Christoffel transform of a discrete minimal surface constitutes a discrete sphere. This fact has been established by Bobenko & Pinkall (1996) for *discrete minimal isothermic surfaces*. However, it is evident that any discrete minimal surface as defined in §3 is discrete isothermic.

5.2.3 Surfaces of constant discrete mean curvature and a discrete Bonnet theorem

In Schief & Konopelchenko (2000), it has been shown that classical surfaces of constant mean curvature constitute O surfaces. It turns out that the *discrete isothermic constant mean curvature surfaces* recorded in Bobenko & Pinkall (1996) are indeed discrete O surfaces and, in fact, coincide with the class of constant discrete mean curvature surfaces defined in §3. Thus, any surface of constant discrete mean curvature is in fact discrete isothermic as in the

differential-geometric context. In order to establish this result, we first observe that any set of parallel discrete O surfaces Σ_κ gives rise to an infinite number of parallel discrete O surfaces by taking linear combinations of the associated position vectors $\vec{R}_\kappa$. For instance, if Σ_1 and Σ_2 are two discrete isothermic surfaces related by the discrete Christoffel transformation then the discrete surfaces $\Sigma_\pm$ with position vectors

$$\vec{R}_\pm = \frac{1}{2}(\vec{R}_2 \pm \vec{R}_1) \tag{44}$$

constitute discrete O surfaces which are parallel to both Σ_1 and Σ_2. The corresponding solutions of the adjoint system $(5)_{2,4}$ are given by

$$H_\pm = \frac{1}{2}(H_2 \pm H_1), \quad K_\pm = \frac{1}{2}(K_2 \pm K_1). \tag{45}$$

Accordingly, the discrete Gaußian curvatures of the discrete surfaces $\Sigma_\pm$ take the form

$$\mathsf{K}_\pm = \frac{H_{\circ(2)}K_\circ + K_{\circ(1)}H_\circ}{H_{\pm(2)}K_\pm + K_{\pm(1)}H_\pm} = \frac{4(H_{\circ(2)}K_\circ + K_{\circ(1)}H_\circ)}{H_{1(2)}K_1 + K_{1(1)}H_1 + H_{2(2)}K_2 + K_{2(1)}H_2} \tag{46}$$

by virtue of (37) and hence coincide. This is not surprising since the transition from (Σ_1, Σ_2) to (Σ_+, Σ_-) may be interpreted at the level of the matrix S as a similarity transformation mapping the case (36) to the case (33).

If we now identify the discrete surface Σ_- with a spherical representation $\Sigma_\circ$ of the discrete isothermic surfaces, that is

$$\vec{R}_- = \vec{R}_\circ, \quad H_- = H_\circ, \quad K_- = K_\circ, \tag{47}$$

then

$$\mathsf{K}_\pm = 1, \quad \mathsf{M}_1 = 1, \quad \mathsf{M}_2 = -1. \tag{48}$$

The latter relations encapsulate a discrete version of a classical theorem due to Bonnet (Eisenhart 1960):

With any surface Σ_+ of constant discrete Gaußian curvature $\mathsf{K}_+ = 1$ one may associate two parallel surfaces Σ_1 and Σ_2 of constant discrete mean curvature $\mathsf{M}_1 = 1$ and $\mathsf{M}_2 = -1$ respectively with position vectors

$$\vec{R}_1 = \vec{R}_+ - \vec{R}_\circ, \quad \vec{R}_2 = \vec{R}_+ + \vec{R}_\circ. \tag{49}$$

This is in agreement with a result presented in Bobenko & Pinkall (1999) and may be regarded as a special case of the following statement:

If the discrete Gaußian curvatures of two parallel curvature lattices $\Sigma_\pm$ are equal, that is $\mathsf{K}_+ = \mathsf{K}_-$ for corresponding quadrilaterals, then the parallel lattices Σ_1 and Σ_2 defined by

$$\vec{R}_1 = \vec{R}_+ - \vec{R}_-, \quad \vec{R}_2 = \vec{R}_+ + \vec{R}_- \tag{50}$$

constitute discrete isothermic surfaces which are related by the discrete Christoffel transformation.

Conversely, any surface Σ_1 of constant discrete mean curvature $\mathsf{M}_1 = 1$ constitutes a discrete isothermic surface and is associated with two particular parallel discrete surfaces, namely a second surface Σ_2 of constant discrete mean curvature $\mathsf{M}_2 = -1$ and a 'middle' surface Σ_+ of constant positive discrete Gaußian curvature $\mathsf{K}_+ = 1$. It is interesting to note that the surfaces Σ_1, Σ_2 and Σ_+ are at constant 'distance', that is $|\vec{R}_1 - \vec{R}_+| = |\vec{R}_2 - \vec{R}_+| = 1$, $|\vec{R}_2 - \vec{R}_1| = 2$.

5.2.4 Discrete 'linear' Weingarten surfaces

Surfaces of constant Gaußian or mean curvature represent particular examples of *Weingarten surfaces* (Eisenhart 1960), that is surfaces in $\mathbb{R}^3$ which admit a functional relation between the principal curvatures. 'Linear' Weingarten surfaces are those corresponding to a functional relation of the form

$$\alpha\mathsf{K} + \beta\mathsf{M} = \gamma, \tag{51}$$

where α, β and γ are arbitrary constants. If Σ constitutes a *discrete linear Weingarten surface*, that is a curvature lattice subject to the constraint (51) with respect to a spherical representation $\Sigma_\circ$, then, on use of the expressions (18) and (23) for the discrete Gaußian and mean curvatures K and M respectively, the above relation may be brought into the form

$$\underline{H}_{(2)} \cdot \underline{K} + \underline{K}_{(1)} \cdot \underline{H} = 0, \quad S = \begin{pmatrix} \gamma & \beta \\ \beta & -\alpha \end{pmatrix} \tag{52}$$

with the identification

$$\underline{H} = (H, H_\circ), \quad \underline{K} = (K, K_\circ). \tag{53}$$

Thus, discrete linear Weingarten surfaces constitute discrete O surfaces which are parallel to surfaces of discrete constant Gaußian curvature since the matrix S as given by $(52)_2$ may be mapped by means of an appropriate similarity transformation to either (30) or (33) provided that $\det S \neq 0$. At the level of the position matrix $\underline{\vec{R}}$, this corresponds to a linear transformation.

5.2.5 Discrete Guichard surfaces

The class of discrete O surfaces in $\mathbb{R}^3$ corresponding to a three-dimensional dual space endowed with the indefinite metric

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{54}$$

evidently includes discrete isothermic surfaces. On the other hand, in Schief & Konopelchenko (2000), it has been shown how classical *Guichard surfaces* (Eisenhart 1962) may be retrieved in the differential-geometric setting by confining Σ_3 to the unit sphere. Accordingly, in the present context, if Σ_3 is taken to be a spherical representation of Σ_1 and Σ_2 then the latter constitute *discrete Guichard surfaces*.

5.2.6 Discrete Petot surfaces

Another canonical class of discrete O surfaces is obtained by identifying the three-dimensional Euclidean space with its dual. Thus, if we set

$$\underline{H} = \vec{X}^{\mathsf{T}}, \quad \underline{K} = \vec{Y}^{\mathsf{T}}, \quad p = q \tag{55}$$

then the linear systems (25) coincide and the discrete orthogonality condition $(9)_2$ reduces to

$$\vec{X} \cdot \vec{Y} + p = 0, \tag{56}$$

leading to

$$\vec{X}_{(2)} \cdot \vec{Y} = 0, \quad \vec{Y}_{(1)} \cdot \vec{X} = 0. \tag{57}$$

Alternatively, the identification

$$\underline{H} = \vec{X}^{\mathsf{T}}, \quad \underline{K} = -\vec{Y}^{\mathsf{T}}, \quad p = -q \tag{58}$$

results in

$$\vec{X} \cdot \vec{Y} = 0, \quad \vec{X}_{(2)} \cdot \vec{Y}_{(1)} = 0. \tag{59}$$

The characterizations (57) or (59), which, in geometric terms, express the fact that there exist two right angles in each quadrilateral, have been used to define *discrete Petot surfaces* (Schief 1997). These represent the constituent members of discrete Darboux-Egorov-type triply orthogonal systems of surfaces. It is noted, however, that the above discrete Petot O surfaces are not generic due to the particular form $(55)_{1,2}$ or $(58)_{1,2}$ of the adjoint eigenfunctions $\underline{H}$ and $\underline{K}$. The class of discrete Petot O surfaces nevertheless enshrines the generic class of discrete Petot surfaces in the sense that any discrete Petot surface may be obtained from a discrete Petot O surface by application of an appropriate discrete Combescure transformation.

6 A Bäcklund transformation for discrete O surfaces

The transformation theory of conjugate lattices is by now well-established (Konopelchenko & Schief 1998; Doliwa *et al.* 1997). Here, we focus on the natural discrete analogue of the classical Fundamental transformation (Eisenhart 1962). Since the *discrete Fundamental transformation* (Konopelchenko & Schief 1998) commutes with the discrete Combescure transformation, it may be simultaneously applied to sets of parallel conjugate lattices.

6.1 The discrete Fundamental and Ribaucour transformations

The discrete Fundamental transformation is generated by two pairs of scalar solutions of the linear systems (25) and corresponding bilinear potentials of the

form (26). Thus, for a given pair of functions p, q associated with a set of parallel conjugate lattices Σ_κ, let $\{X, Y\}$ and $\{H, K\}$ be solutions of the linear systems

$$\begin{aligned} X_{(2)} &= \frac{X + qY}{\Gamma}, \quad H_{(2)} = \frac{H + pK}{\Gamma} \\ Y_{(1)} &= \frac{Y + pX}{\Gamma}, \quad K_{(1)} = \frac{K + qH}{\Gamma}. \end{aligned} \tag{60}$$

In the sequel, we refer to X, Y and H, K as *eigenfunctions* and *adjoint eigenfunctions* respectively. Three bilinear potentials $\vec{M}, \underline{M}$ and M may now be introduced according to

$$\begin{aligned} \vec{M}_{(1)} - \vec{M} &= \vec{X}H, \quad \underline{M}_{(1)} - \underline{M} = X\underline{H}, \quad M_{(1)} - M = XH \\ \vec{M}_{(2)} - \vec{M} &= \vec{Y}K, \quad \underline{M}_{(2)} - \underline{M} = Y\underline{K}, \quad M_{(2)} - M = YK. \end{aligned} \tag{61}$$

A second set of parallel conjugate lattices Σ'_κ is now obtained as follows (Konopelchenko & Schief 1998):

Theorem 1 (The discrete Fundamental transformation) *The linear systems (25) and the defining relations (26) are invariant under*

$$(\vec{\underline{R}}, \vec{X}, \vec{Y}, \underline{H}, \underline{K}, p, q) \to (\vec{\underline{R}}', \vec{X}', \vec{Y}', \underline{H}', \underline{K}', p', q'), \tag{62}$$

where

$$\vec{\underline{R}}' = \vec{\underline{R}} - \frac{\vec{M}\underline{M}}{M} \tag{63}$$

and

$$\begin{aligned} \vec{X}' &= \sqrt{\frac{M}{M_{(1)}}}\left(\vec{X} - \frac{X\vec{M}}{M}\right), \qquad \vec{Y}' = \sqrt{\frac{M}{M_{(2)}}}\left(\vec{Y} - \frac{Y\vec{M}}{M}\right) \\ \underline{H}' &= \sqrt{\frac{M}{M_{(1)}}}\left(\underline{H} - \frac{H\underline{M}}{M}\right), \qquad \underline{K}' = \sqrt{\frac{M}{M_{(2)}}}\left(\underline{K} - \frac{K\underline{M}}{M}\right) \\ p' &= \sqrt{\frac{M^2}{M_{(1)}M_{(2)}}}\left(p - \frac{YH}{M}\right), \qquad q' = \sqrt{\frac{M^2}{M_{(1)}M_{(2)}}}\left(q - \frac{XK}{M}\right). \end{aligned} \tag{64}$$

In order to verify the above transformation laws, it is convenient to be aware of the relation

$$\Gamma'^2 = \frac{MM_{(12)}}{M_{(1)}M_{(2)}}\,\Gamma^2. \tag{65}$$

It is emphasized that the above transformation may also be regarded as a mapping between the sets of discrete dual surfaces Σ^k and Σ'^k.

If the conjugate lattices Σ_κ are discrete orthogonal, that is the condition $(9)_2$ is satisfied, then it is readily verified that the quantities

$$X = (\vec{M}_{(1)} + \vec{M}) \cdot \vec{X}, \quad Y = (\vec{M}_{(2)} + \vec{M}) \cdot \vec{Y} \tag{66}$$

constitute particular eigenfunctions. This choice of eigenfunctions in the definitions of the bilinear potentials $\vec{M}$ and M leads, in turn, to the relations

$$\vec{M}_{(1)}^2 - \vec{M}^2 = M_{(1)} - M, \quad \vec{M}_{(2)}^2 - \vec{M}^2 = M_{(2)} - M \tag{67}$$

so that we may set

$$\vec{M}^2 = M. \tag{68}$$

It is now straightforward to show that

$$\vec{X}'^2 = 1, \quad 2\vec{X}' \cdot \vec{Y}' + p' + q' = 0, \quad \vec{Y}'^2 = 1. \tag{69}$$

Thus, it turns out that curvature lattices and the normalization (11) are preserved by the discrete Fundamental transformation if the eigenfunctions X, Y and the bilinear potential M are chosen to be (66) and (68) respectively. Under these circumstances, the discrete Fundamental transformation becomes the *discrete Ribaucour transformation* as set down in Konopelchenko & Schief (1998). Indeed, in the natural continuum limit, the above particular discrete Fundamental transformation reduces to the classical Ribaucour transformation for surfaces in $\mathbb{R}^3$ parametrized in terms of curvature coordinates (Eisenhart 1962).

6.2 Application to discrete O surfaces

It is remarkable that the discrete Ribaucour transformation may be constrained in such a way that discrete orthogonality of the dual conjugate lattices is also sustained. In fact, as a by-product, a parameter-dependent linear representation of discrete O surfaces is obtained. As in the preceding, we first observe that the quantities

$$H = \lambda(\underline{M}_{(1)} + \underline{M}) \cdot \underline{H}, \quad K = \lambda(\underline{M}_{(2)} + \underline{M}) \cdot \underline{K} \tag{70}$$

constitute particular adjoint eigenfunctions. The constant parameter λ is now non-trivial as we have already specified the eigenfunctions X and Y. The associated potentials $\underline{M}$ and M then obey the relations

$$\lambda(\underline{M}_{(1)}^2 - \underline{M}^2) = M_{(1)} - M, \quad \lambda(\underline{M}_{(2)}^2 - \underline{M}^2) = M_{(2)} - M \tag{71}$$

so that

$$\lambda \underline{M}^2 = M \tag{72}$$

is, at least, consistent. It is shown below that this constraint is indeed admissible. On this assumption, we now proceed and note that

$$\underline{H}'^2 = \underline{H}^2, \quad 2\underline{H}' \cdot \underline{K}' + q'\underline{H}'^2 + p'\underline{K}'^2 = 0, \quad \underline{K}'^2 = \underline{K}^2, \tag{73}$$

which implies that the transformed dual conjugate lattices Σ'^k are also discrete orthogonal with any normalization of the form

$$\underline{H}^2 = \alpha(n_1), \quad \underline{K}^2 = \beta(n_2) \tag{74}$$

unchanged (cf. (29)).

Finally, insertion of the (adjoint) eigenfunctions X, Y and H, K as given by (66) and (70) respectively into the defining relations (61) produces the following *Lax pair* for discrete O surfaces:

Theorem 2 (A Lax pair for discrete O surfaces) *The linear system*

$$\begin{aligned} \begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix}_{(1)} - \begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix} &= \frac{2}{1-\alpha\lambda} \begin{pmatrix} \alpha\lambda\vec{X}\vec{X}^{\intercal} & \lambda\vec{X}\underline{H} \\ \underline{H}^{\intercal}\vec{X}^{\intercal} & \lambda\underline{H}^{\intercal}\underline{H} \end{pmatrix} \begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix} \\ \begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix}_{(2)} - \begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix} &= \frac{2}{1-\beta\lambda} \begin{pmatrix} \beta\lambda\vec{Y}\vec{Y}^{\intercal} & \lambda\vec{Y}\underline{K} \\ \underline{K}^{\intercal}\vec{Y}^{\intercal} & \lambda\underline{K}^{\intercal}\underline{K} \end{pmatrix} \begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix} \end{aligned} \tag{75}$$

is compatible modulo the linear systems (25) and the discrete orthogonality conditions (9) and (28). It admits the first integral

$$\vec{M}^2 - \lambda\underline{M}^2 = \text{const.} \tag{76}$$

The existence of the first integral (76) guarantees that the constraint (72) is admissible. Consequently, we are now in a position to formulate the following theorem:

Theorem 3 (A Bäcklund transformation for discrete O surfaces) *Let $\underline{\vec{R}}$ be the position matrix of a set of parallel discrete O surfaces Σ_κ and their duals Σ^k and $\vec{X}, \vec{Y}, \underline{H}, \underline{K}$ corresponding tangent vectors. If the vectors $\vec{M}$ and $\underline{M}$ constitute a solution of the linear system (75) subject to the admissible constraint*

$$\vec{M}^2 = \lambda\underline{M}^2 = M \tag{77}$$

and the scalar M is defined by the latter then the position matrix of a second set of discrete O surfaces $\Sigma'_\kappa, \Sigma'^k$ is given by

$$\underline{\vec{R}}' = \underline{\vec{R}} - \frac{\vec{M}\underline{M}}{M}. \tag{78}$$

We conclude this section with the following important observation. If a discrete O surface Σ_n is identified with a spherical representation of the remaining discrete O surfaces Σ_κ then

$$\vec{R}_n^2 = 1 \tag{79}$$

and it is readily verified that

$$M_n = 2\vec{R}_n \cdot \vec{M} \tag{80}$$

is another admissible constraint. Consequently, the nth component of the transformation law (78) may be cast into the form

$$\vec{R}'_n = \left(\mathbb{1} - 2\frac{\vec{M}\vec{M}^{\intercal}}{\vec{M}^2}\right)\vec{R}_n \tag{81}$$

which implies that

$$\vec{R}'^2_n = 1. \tag{82}$$

Hence, we come to the important conclusion that the above Bäcklund transformation acts within specific sub-classes of discrete O surfaces such as surfaces

of constant discrete Gaußian curvature, discrete minimal or discrete Guichard surfaces. Moreover, it is readily shown that constraints of the form

$$\left(\sum_{\kappa=1}^{n} c_\kappa \vec{R}_\kappa\right)^2 = 1, \tag{83}$$

which generalize (79), may also be preserved. In particular, the specialization (47) leading to constant discrete mean curvature surfaces proves invariant.

7 The discrete pseudosphere and discrete breather pseudospherical surfaces

We conclude this paper with an illustration of the Bäcklund transformation for discrete O surfaces and consider the particular case (30) of discrete pseudospherical surfaces. Thus, we here regard a straight polygon as a (degenerate) seed discrete pseudospherical surface Σ_1 together with an associated 'spherical representation' Σ_2 represented by

$$\vec{R}_1 = \begin{pmatrix} 0 \\ 0 \\ \epsilon n \end{pmatrix}, \quad \vec{R}_2 = \begin{pmatrix} -\sin\nu m \\ \cos\nu m \\ 0 \end{pmatrix} \tag{84}$$

so that the tangent vectors to Σ_1, Σ_2 and their duals read

$$\vec{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{Y} = \begin{pmatrix} \cos(\nu m + \nu/2) \\ \sin(\nu m + \nu/2) \\ 0 \end{pmatrix}, \quad \underline{H} = (\epsilon\ \ 0), \quad \underline{K} = (0\ \ -\delta) \tag{85}$$

with

$$(n,m) = (n_1, n_2), \quad \delta = 2\sin\nu/2. \tag{86}$$

It is evident that the linear systems (25) with $p = q = 0$ and the orthogonality conditions $\vec{X}\cdot\vec{Y} = \underline{H}\cdot\underline{K} = 0$ are satisfied. Accordingly, the linear system (75) for these particular discrete O surfaces becomes

$$\begin{aligned}
\begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix}_{(1)} - \begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix} &= \frac{2\epsilon}{1-\epsilon^2\lambda} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda\epsilon & \lambda & 0 \\ 0 & 0 & 1 & \lambda\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix} \\
\begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix}_{(2)} - \begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix} &= \frac{2\delta}{1-\delta^2\lambda} \begin{pmatrix} \lambda\delta(Y^1)^2 & \lambda\delta Y^1 Y^2 & 0 & 0 & -\lambda Y^1 \\ \lambda\delta Y^1 Y^2 & \lambda\delta(Y^2)^2 & 0 & 0 & -\lambda Y^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -Y^1 & -Y^2 & 0 & 0 & \lambda\delta \end{pmatrix} \begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix}.
\end{aligned} \tag{87}$$

The latter decouples into two systems of linear *non-autonomous* ordinary difference equations for $M^3(n), M_1(n)$ and $M^1(m), M^2(m), M_2(m)$ respectively. The constants of integration in the general solution of (87) have to be chosen in such a way that the admissible constraints (77) and $(80)_{n=2}$ are satisfied.

In the differential-geometric context, that is in the natural continuum limit

$$x = \epsilon n, \quad y = \nu m, \quad \epsilon, \nu \to 0, \tag{88}$$

it has been shown (Schief & Konopelchenko 2000) that the Bäcklund transformation for O surfaces produces either Beltrami's classical *pseudosphere* (Eisenhart 1960) or 'stationary' *breather pseudopsherical surfaces* (Rogers & Schief 2000). The pseudosphere and a particular breather pseudospherical surface with $\mathbb{Z}_6$ rotational symmetry is depicted in Figure 3. It is therefore expected that the

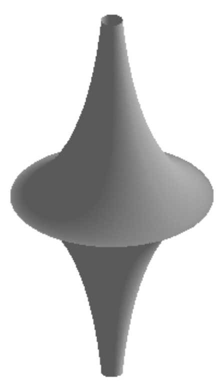

Figure 2: The classical pseudosphere

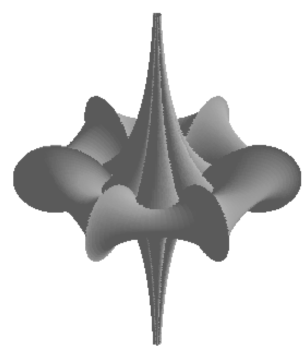

Figure 3: A breather pseudospherical surface

Bäcklund transformation for discrete O surfaces delivers discretizations of the pseudosphere and breather pseudospherical surfaces. For brevity, we here only state that a careful analysis of the solution of the linear system (87) and the associated Bäcklund transformation leads to the following result:

In the case $\lambda = 1/4$, the position vector $\vec{R}_1'$ of the Bäcklund transform Σ_1' may be reduced to

$$\vec{R}_1' = \begin{pmatrix} \dfrac{\sin \nu m}{\cosh \tau n} \\ -\dfrac{\cos \nu m}{\cosh \tau n} \\ \epsilon n - \tanh \tau n \end{pmatrix} \tag{89}$$

modulo translations of the form $n \to n + \text{const}$, $m \to m + \text{const}$, where

$$\tau = \ln\left(\frac{2+\epsilon}{2-\epsilon}\right). \tag{90}$$

This discrete pseudsospherical surface of 'revolution' is nothing but a discretization of Beltrami's classical pseudosphere. The parameters ϵ and ν, which may be chosen arbitrarily, constitute measures of the 'quality' of the discretization. If ν is rational then the discrete pseudosphere admits a discrete rotational symmetry. Two such discrete pseudospheres are displayed in Figure 4.

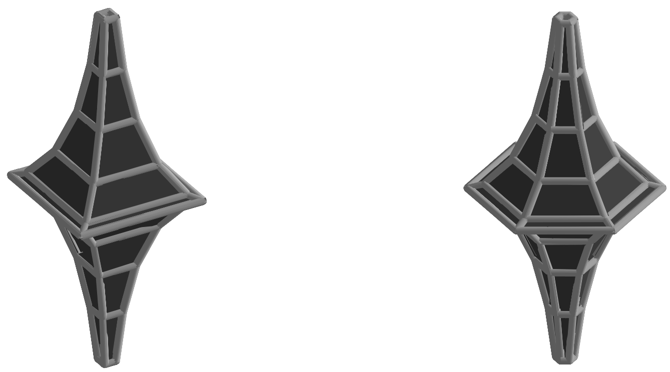

Figure 4: Discrete pseudospheres

If $\lambda \neq 1/4$ then integration of the linear system (87) and specification of the constants of integration lead to the position vector

$$\vec{R}_1' = \begin{pmatrix} 0 \\ 0 \\ \epsilon n \end{pmatrix} - \frac{2d}{c}\frac{\cosh \tau n}{c^2 \sin^2 \kappa m + d^2 \cosh^2 \tau n} \times \begin{pmatrix} -\sin\kappa m \sin\nu m - d\cos\kappa m\cos\nu m \\ \sin\kappa m\cos\nu m - d\cos\kappa m\sin\nu m \\ d\sinh\tau n \end{pmatrix}, \tag{91}$$

where

$$\lambda = \frac{c^2}{4}, \quad c^2 + d^2 = 1, \quad \tau = \ln\left(\frac{2+\epsilon c}{2-\epsilon c}\right), \quad \kappa = 2\arctan\left(d\tan\frac{\nu}{2}\right). \tag{92}$$

These discrete pseudospherical surfaces indeed constitute discretizations of the above-mentioned breather pseudospherical surfaces. If the constants c and d are real then there exists a discrete rotational symmetry if κ/ν is rational, that is

$$\frac{\kappa}{\nu} = \frac{\mathsf{p}}{\mathsf{q}}, \quad \mathsf{p}, \mathsf{q} \in \mathbb{Z}. \tag{93}$$

A variety of *discrete breather pseudospherical surfaces* corresponding to different choices of p and q may now be generated. In Figure 5, several discretizations

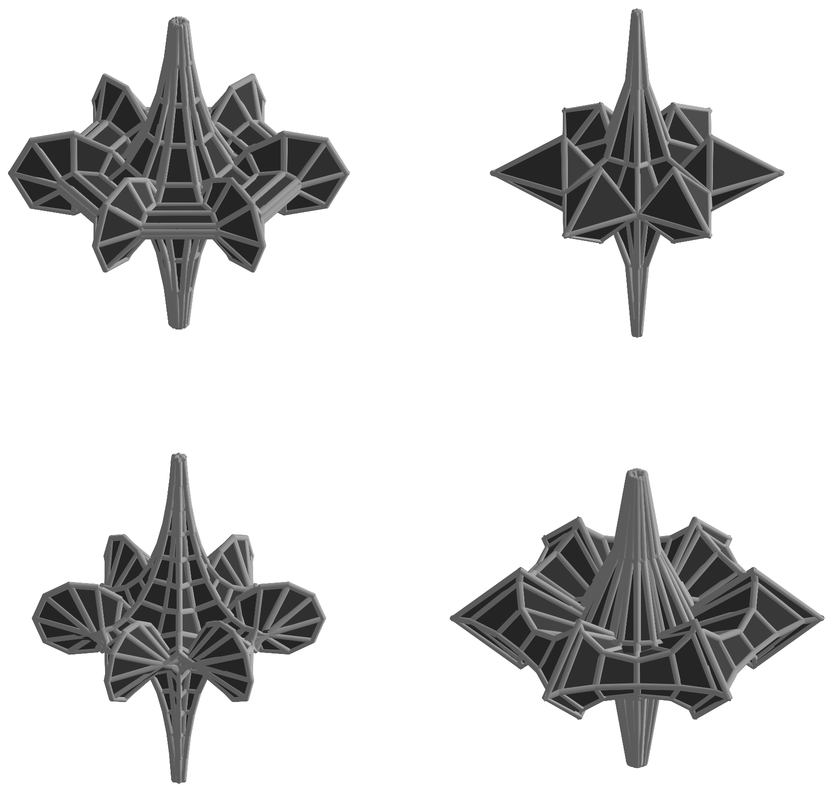

Figure 5: Discrete breather pseudospherical surfaces for $\frac{\mathsf{p}}{\mathsf{q}} = \frac{3}{4}$

of the breather pseudospherical surface displayed in Figure 3 characterized by $\mathsf{p}/\mathsf{q} = 3/4$ are shown.

As in the differential-geometric context, it is interesting to note that the Bäcklund transformation for discrete O surfaces does not reduce to a discrete version of the classical Bäcklund transformation for pseudospherical surfaces. In fact, as discussed above, a single application of the Bäcklund transformation for discrete O surfaces to a straight polygon produces discrete pseudospheres or discrete breather pseudospherical surfaces. By contrast, a single application of the classical Bäcklund transformation to a straight line results in a one-parameter family of *Dini surfaces* including the pseudosphere (Eisenhart 1960). A second application then leads to breather pseudospherical surfaces if one assumes that the two Bäcklund parameters are complex conjugates (Rogers & Schief 2000). It is also emphasized that the procedure for the generation of discrete breather pseudospherical surfaces outlined here involves the solution of

a system of *non-autonomous* difference equations.

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